

A NON-STRICTLY PSEUDOCONVEX DOMAIN FOR WHICH THE SQUEEZING FUNCTION TENDS TO ONE TOWARDS THE BOUNDARY

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ABSTRACT. In recent work by Zimmer it was proved that if $\Omega \subset \mathbb{C}^n$ is a bounded convex domain with C^∞ -smooth boundary, then Ω is strictly pseudoconvex provided that the squeezing function approaches one as one approaches the boundary. We show that this result fails if Ω is only assumed to be C^2 -smooth.

1. INTRODUCTION

We recall the definition of the squeezing function $S_\Omega(z)$ on a bounded domain $\Omega \subset \mathbb{C}^n$. If $z \in \Omega$, and $f_z : \Omega \rightarrow \mathbb{B}^n$ is an embedding with $f_z(z) = 0$, we set

$$(1.1) \quad S_{\Omega, f_z}(z) := \sup\{r > 0 : B_r(0) \subset f_z(\Omega)\},$$

and then

$$(1.2) \quad S_\Omega(z) := \sup_{f_z} \{S_{\Omega, f_z}(z)\}.$$

A guiding question is the following: which complex analytic properties of Ω are encoded by the behaviour of S_Ω ? For instance, if S_Ω is bounded away from zero, then Ω is necessarily pseudoconvex, and the Kobayashi-, Carathéodory-, Bergman- and the Kähler-Einstein metric are complete, and they are pairwise quasi-isometric (see [8]). Recently, Zimmer [9] proved that if

$$(1.3) \quad \lim_{z \rightarrow b\Omega} S_\Omega(z) = 1$$

for a C^∞ -smooth, bounded convex domain, then Ω is necessarily strictly pseudoconvex. In this short note we will show that this does not hold for C^2 -smooth domains.

Theorem 1.1. *There exists a bounded convex C^2 -smooth domain $\Omega \subset \mathbb{C}^n$ which is not strongly pseudoconvex, but*

$$(1.4) \quad \lim_{z \rightarrow b\Omega} S_\Omega(z) = 1,$$

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where $S_\Omega(z)$ denotes the squeezing function on Ω .

For further results about the squeezing function the reader may also consult the references [1], [2],[3],[4],[5],[6],[7], [8], [9]. In the last section we will post some open problems.

2. THE CONSTRUCTION

2.1. The construction in \mathbb{R}^n and curvature estimates. We start by describing a construction of a convex domain Ω in \mathbb{R}^n with a single non-strictly convex point. Afterwards we will explain how to make the construction give the conclusion of Theorem 1.1 for each $n = 2m$, when we make the identification with \mathbb{C}^m .

Let $x = x_1, \dots, x_n$ denote the coordinates on \mathbb{R}^n . For any $k \in \mathbb{N}$ we let B_k denote the ball

$$(2.1) \quad B_k := \{x \in \mathbb{R}^n : x_1^2 + \dots + x_{n-1}^2 + (x_n - k)^2 < k^2\}.$$

On some fixed neighbourhood of the origin, each boundary bB_k may be written as a graph of a function

$$(2.2) \quad x_n = \psi_k(x') = \psi_k(x_1, \dots, x_{n-1}) = k - \sqrt{k^2 - |x'|^2} = \frac{1}{2k}|x'|^2 + \text{h.o.t.}$$

Fix a smooth cut-off function $\chi(x') = \chi(|x'|)$ with compact support in $\{|x'| < 1\}$ which is one near the origin. We will create a new limit graphing function $f(x')$ by subsequently gluing the functions ψ_k and ψ_{k+1} by setting

$$(2.3) \quad g_k(x') = \psi_k(x') + \chi\left(\frac{x'}{\epsilon_k}\right)(\psi_{k+1}(x') - \psi_k(x')),$$

where the sequence ϵ_k will converge rapidly to zero, and the boundary of our domain Ω will be defined (locally) as the graph Σ of the function f defined as follows: start by setting $f_k := \psi_k$ for some $k \in \mathbb{N}$. Then define f_{k+1} inductively by setting $f_{k+1} = f_k$ for $\|x'\| \geq \epsilon_k$ and then $f_{k+1} = g_k$ for $\|x'\| < \epsilon_k$. Finally we set $f = \lim_{k \rightarrow \infty} f_k$.

To ensure that Ω is convex we will need to estimate the curvature of Σ , and estimates of the curvature of the partial graphs $\Sigma_k = \{x, g_k(x)\}$ will be necessary to prove Theorem 1.1. Informally our goal is to show the following: *There exist $N, m \in \mathbb{N}, N > m$, such that if $k \geq N$ and if ϵ_k is sufficiently small (depending on k), then Σ_k curves, at every point and in all directions, more than bB_{k+m} and less than bB_{k-m} .*

We make this more precise. The surface Σ_k has a defining function $\rho_k(x) = g_k(x') - x_n$. If v_p is a tangent vector to Σ_k at $p = (x', g_k(x'))$, the curvature of Σ_k in the direction of v_p is defined as

$$(2.4) \quad \kappa_p^{\Sigma_k}(v_p) := \frac{H\rho_k(p)(v_p)}{\|\nabla\rho_k(p)\| \|v_p\|^2},$$

where $\nabla\rho_k$ is the gradient, and $H\rho_k$ is the Hessian of ρ_k (which is equal to the Hessian of g_k). The curvature (2.4) depends only on the direction of v_p ,

and the curvature of bB_k is $\frac{1}{k}$ at all points and in all directions. The precise statement of our goal stated above is

Lemma 2.1. *Let ψ_k and χ be defined as above for $k \in \mathbb{N}$. There exist $N, m \in \mathbb{N}, N > m$, such that if each ϵ_k is sufficiently small (depending on k), and $k \geq N$, then*

$$(2.5) \quad \frac{1}{k+m} \leq \kappa_p^{\Sigma_k}(v_p) \leq \frac{1}{k-m},$$

for all v_p tangent to Σ_k .

It is now easy to see that if $\epsilon_k \searrow 0$ sufficiently fast, then Ω is convex, and strictly convex away from the origin. If we let Ω_k denote the domain whose boundary near the origin is given by the graph of f_k , we see that Ω_k is strictly convex, the Hessian being positive definite everywhere. Moreover $\Omega = \cup_k \Omega_k$, and so Ω is convex.

Proof. (of Lemma 2.1) When we estimate the curvature we may assume that the functions g_k are simply

$$(2.6) \quad g_k(x') = \psi_k(x') - \chi\left(\frac{x'}{\epsilon_k}\right)\left(\frac{1}{2k(k+1)}\right)|x'|^2 =: \psi_k(x') + \sigma_k(x'),$$

since the higher order terms missing in this expression of g_k can be made insignificant by choosing ϵ_k small enough. Because of the $|x'|^2$ -term it is easy to see that

$$(2.7) \quad dg_k(x') = d\psi_k(x') + \triangle_k(x'),$$

and

$$(2.8) \quad Hg_k(x') = H\psi_k(x') + h_k(x'),$$

where the coefficients in both \triangle_k and h_k are of order of magnitude $\frac{1}{k^2}$ independently of k and of the choice of a small ϵ_k .

Fix a point x' and a vector $v \in \mathbb{R}^{n-1}$ with $\|v\| = 1$. Then a tangent vector v_p at the point $(x', g_k(x'))$ is given by

$$(2.9) \quad v_p = (v, dg_k(x')(v)) = (v, d\psi_k(x')(v) + \triangle_k(x')(v)).$$

Estimating the curvature we see that

$$\begin{aligned}
\kappa_p^{\Sigma_k}(v_p) &= \frac{(H\psi_k(x') + h_k(x'))(v_p)}{\|\nabla\rho_k(p)\|\|v_p\|^2} \\
&= \frac{(H\psi_k(x'))((v, d\psi_k(x')v) + (0', \Delta_k(x')(v)))}{\|-\mathbf{e}_n + \nabla\psi_k(p) + \nabla\sigma_k(x')\|\|(v, d\psi_k(x')(v)) + (0', \Delta_k(x'))\|^2} \\
&\quad + O\left(\frac{1}{k^2}\right) \\
&= \frac{(H\psi_k(x'))((v, d\psi_k(x')v))}{\|-\mathbf{e}_n + \nabla\psi_k(x')\|(1 + O(\frac{1}{k^2}))\|(v, d\psi_k(x')(v))\|^2(1 + O(\frac{1}{k^2}))^2} \\
&\quad + O\left(\frac{1}{k^2}\right) \\
&= \frac{(H\psi_k(x'))((v, d\psi_k(x')v))}{\|-\mathbf{e}_n + \nabla\psi_k(x')\|\|(v, d\psi_k(x')(v))\|^2} + O\left(\frac{1}{k^2}\right) \\
&= \frac{1}{k} + O\left(\frac{1}{k^2}\right),
\end{aligned}$$

where the term $\frac{1}{k}$ comes from the fact that the expression above is the formula for the curvature of a ball of radius k . From this it is straightforward to deduce the existence of an m such that the lemma holds. \square

2.2. The squeezing function on Ω . We will now explain why the squeezing function goes to one uniformly as we approach $b\Omega$ provided that the ϵ_k 's decrease sufficiently fast. Let N, m be as in Lemma 2.1, and start by setting $f_k = \psi_k$ for some $k > N$.

Fix some small $\delta_k > 0$. By Lemma 2.1, if ϵ_k is small enough, we can for each $p = (x', x_n) \in b\Omega_k$, $\|x'\| < \delta_k$, find a ball B of radius $k + m$ containing Ω_k such that $p \in bB$. By the same lemma we can for each such p also find a local piece of a ball of radius $k - m$ touching p from the inside of Ω_k , and the size of the local ball is uniform. So using Lemma 3.1 we may find a $t_k > 0$ small enough such that

$$(2.10) \quad S_{\Omega_k}(x', x_n) \geq 1 - \frac{m}{(k + m)}$$

if $x_n \leq t_k$.

Next, again by Lemma 2.1, we find a $\delta_{k+1} < \delta_k$ such that if ϵ_{k+1} is small enough, then for each $p = (x', x_n) \in b\Omega_{k+1}$ with $\|x'\| < \delta_{k+1}$, we may oscillate with balls of radius $k + 1 - m$ and $k + 1 + m$ respectively. So there is a $t_{k+1} < t_k$ such that

$$(2.11) \quad S_{\Omega_{k+1}}(x', x_n) \geq 1 - \frac{m}{(k + 1 + m)}$$

if $x_n \leq t_{k+1}$. Furthermore, by further decreasing ϵ_{k+1} we can keep the estimate (2.10) with Ω_k replaced by Ω_{k+1} . The reason is the following. First of all, by [5] there exists a constant C_k such that

$$(2.12) \quad S_{\Omega_k}(z) \geq 1 - C_k \cdot \text{dist}(z, b\Omega_k),$$

and near any compact $K \subset b\Omega_k$ away from 0, this estimate is not going to be disturbed by a small perturbation of $b\Omega_k$ near the point 0; the estimate is obtained by using oscillating balls at points of K whose boundaries will stay bounded away from 0. Furthermore, on any compact subset of Ω_k we have that $S_{\Omega_{k+1}} \rightarrow S_{\Omega_k}$ as $\epsilon_{k+1} \rightarrow 0$.

Continuing in this fashion, we obtain a decreasing sequence $0 < t_j < t_{j+1}$, $j = k, k+1, \dots$, and an increasing sequence of domains Ω_j , such that for each j we have that

$$(2.13) \quad S_{\Omega_j}(x', x_n) \geq 1 - \frac{m}{(k+i+m)}$$

for $t_{k+i} \leq x_n \leq t_{k+i-1}$, for $i \leq j$. The result now follows from Lemma 3.2.

3. LEMMATA

Let $0 < s < 1/2$, $0 < d < r < 1$, and set $B_s = B(s, 1-s)$. Furthermore we set

$$(3.1) \quad B_{s,d} = B_s \cap \{(z_1, z') \in \mathbb{B}^n : \mathcal{R}(z_1) > d\}.$$

Lemma 3.1. *If $B_{s,d} \subset \Omega \subset \mathbb{B}^n$, and if $r > 1 - \frac{sd}{4}$, then $S_\Omega(r, 0) > 1 - s$.*

Proof. Set $\mu = 1 - s$ and $\eta = \frac{d}{2}$, and then

$$(3.2) \quad B_\eta^\mu = \{(z_1, z') \in \mathbb{C}^n : |z_1 - (1 - \eta)|^2 + \frac{\eta}{\mu}|z'|^2 < \eta^2\}.$$

Then certainly $\mathcal{R}(z_1) > d$ on B_η^μ , and we also have that $B_\eta^\mu \subset B_s$. To see the latter, we translate the two balls sending $(1, 0')$ to the origin, where they are defined by

$$(3.3) \quad \tilde{B}_s = \{(z_1, z') : 2\mu\mathcal{R}(z_1) + |z|^2 < 0\},$$

and

$$(3.4) \quad \tilde{B}_\eta^\mu = \{(z_1, z') : 2\eta\mathcal{R}e(z_1) + |z_1|^2 + \frac{\eta}{\mu}|z'|^2 < 0\}.$$

And

$$\begin{aligned} 2\eta\mathcal{R}e(z_1) + |z_1|^2 + \frac{\eta}{\mu}|z'|^2 < 0 &\Rightarrow 2\eta\mathcal{R}e(z_1) + \frac{\eta}{\mu}|z_1|^2 + \frac{\eta}{\mu}|z'|^2 < 0 \\ &\Leftrightarrow 2\mu\mathcal{R}(z_1) + |z|^2 < 0. \end{aligned}$$

According to Lemma 3.5 in [5] we have that

$$(3.5) \quad S_\Omega(r, 0) \geq \sqrt{\mu} \sqrt{1 - 2(1-r)\frac{1}{\eta}} = \sqrt{(1-s)(1 - \frac{4(1-r)}{d})},$$

from which the lemma follows easily. \square

Lemma 3.2. *Let $\Omega_j \subset \Omega_{j+1}$ for $j \in \mathbb{N}$, set $\Omega = \cup_j \Omega_j$, and assume that Ω is bounded. Let $z \in \Omega$, and assume that $S_{\Omega_j}(z) > 1 - \delta$ for all j large enough so that $z \in \Omega_j$. Then $S_\Omega(z) \geq 1 - \delta$.*

Proof. Let $f_j : \Omega_j \rightarrow \mathbb{B}^n$ be an embedding such that $f_j(z) = 0$ and $B_{1-\delta}(0) \subset f_j(\Omega_j)$. By passing to a subsequence we may assume that $f_j \rightarrow f : \Omega \rightarrow \mathbb{B}^n$ u.o.c., with $f(z) = 0$. Setting $g_j = f_j^{-1} : B_{1-\delta}(0) \rightarrow \Omega$ we may also assume that $g_j \rightarrow g$ u.o.c. Then $f|_{g(B_{1-\delta}(0))} = g^{-1}$, from which the result follows. \square

4. SOME OPEN PROBLEMS

Problem 4.1. Does Zimmer's result hold for pseudoconvex domains of class C^∞ ?

Problem 4.2. How much smoothness is needed for Zimmer's result hold for convex/pseudoconvex domains?

Problem 4.3. Let $\Omega \subset \mathbb{C}^2$ be a bounded pseudoconvex domain of class C^∞ . Is $S_\Omega(z)$ bounded away from zero?

Yeung [8] showed that the answer is yes for strongly convex domains in \mathbb{C}^n , and Kim-Zhang [6] and Deng-Guan-Zhang [3] showed that the answer is yes for strictly pseudoconvex domains. On the other hand, Fornæss-Rong [4] showed that the answer is no for $n \geq 3$.

Quantifying the asymptotic behaviour of the squeezing function, Fornæss-Wold [5] showed that

- (i) $S_\Omega(z) \geq 1 - C \text{dist}(z, b\Omega)$, and
- (ii) $S_\Omega(z) \geq 1 - C \sqrt{\text{dist}(z, b\Omega)}$,

for strongly pseudoconvex domains of class C^4 and C^3 respectively. Diederich-Fornæss-Wold [1] showed that if the squeezing function approaches one essentially faster than (i), then Ω is biholomorphic to the unit ball.

Problem 4.4. What is the optimal estimate for the squeezing function for strictly pseudoconvex domains of class C^k with $k < 4$?

Let $\phi : \mathbb{B}^2 \rightarrow \mathbb{C}^2$ be defined as $\phi(z_1, z_2) := (z_1, -z_2 \log(z_1 - 1))$. Then $\Omega := \phi(\mathbb{B}^2)$ is of class C^1 , and $(1, 0)$ is a non-strictly pseudoconvex boundary point of Ω . So S_Ω being identically equal to one does not even imply strict pseudoconvexity in the case of C^1 -smooth boundaries.

Problem 4.5. Let $\phi : \mathbb{B}^n \rightarrow \Omega$ be a biholomorphism, and assume that Ω is a bounded C^2 -smooth domain. Is Ω strictly pseudoconvex?

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